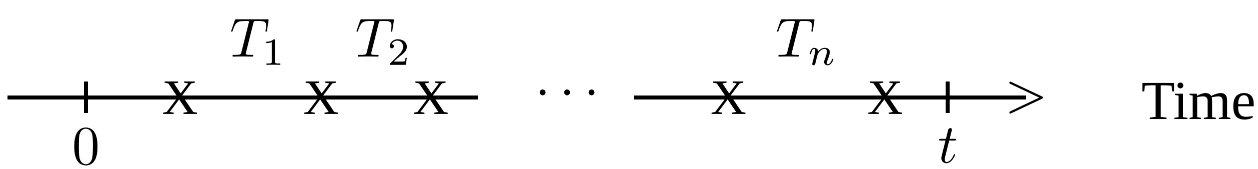
5. General Renewal Processes



5. General Renewal Processes

5.1 Asymptotic Distribution

Event Times: T_i are iid r.v. with pdf $q(t)$ and $Q(t) = \int_t^\infty q(x)dx$

$N(t)$ = No. of events in $(0, t]$

$$S_n = T_1 + T_2 + ... + T_n \quad \text{time to } n^{th} \text{ event}$$

Assume $\quad E(T_i) = m, \qquad V(T_i) = \sigma^2$

$$\boxed{P\{S_n > t\} = P\{N(t) < n\}}$$

Question: What is distribution of $N(t)$ as $t \to \infty$. To find asymptotic distribution n must be allowed to "grow" as t becomes large. Define n_t to depend on t. Then

$$P\{S_{n_t} > t\} = P\{N(t) < n_t\}$$

and we wish to evaluate the above as $t \to \infty$.

$$P\{S_{n_t} > t\} = P\{N(t) < n_t\}$$

Find $\lim_{t\to\infty} P\{S_{n_t} > t\} = \lim_{t\to\infty} P\{N(t) < n_t\}$

By Central limit theorem

$$\frac{S_{n_t} - n_t m}{\sigma\sqrt{n_t}} \sim N(0,1) \text{ as } n_t \to \infty$$

$$P\{S_{n_t} > t\} = P\left\{Y > \frac{t - n_t m}{\sigma\sqrt{n_t}}\right\} \to Q\left(\frac{t - n_t m}{\sigma\sqrt{n_t}}\right) \quad \text{as } n_t \to \infty$$

where $\quad Q(z) = \int_z^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$

Let $\quad \boxed{n_t = \frac{t}{m} + y\sigma\sqrt{t/m^3}}$

Then $\quad t - n_t m = t - m\left[\frac{t}{m} + y\sigma\sqrt{t/m^3}\right] = -y\sigma\sqrt{t/m}$

$$\frac{t - n_t m}{\sigma\sqrt{n_t}} = \frac{-y\sqrt{t/m}}{\left\{\frac{t}{m}\left[1 + y\sigma/\sqrt{tm}\,\right]\right\}^{1/2}} = \frac{-y}{\left[1 + y\sigma/\sqrt{tm}\right]^{1/2}}$$

and as $t \to \infty$ $$\frac{t - n_t m}{\sigma\sqrt{n_t}} \to -y$$

and $$\boxed{\lim_{t\to\infty} P\{S_{n_t} > t\} = Q(-y)}$$

$$n_t = \frac{t}{m} + y\sigma\sqrt{t/m^3}$$

$$\lim_{t\to\infty} P\{S_{n_t} > t\} = \lim_{t\to\infty} P\{N(t) < n_t\} = Q(-y)$$

$$P\{N(t) < n_t\} = P\left\{\frac{N(t) - t/m}{\sigma\sqrt{t/m^3}} < y\right\} \to Q(-y) \text{ as } t \to \infty$$

or since $P(y) = Q(-y)$ for normal distribution

$$\lim_{t\to\infty} P\left\{\frac{N(t)-t/m}{\sigma\sqrt{t/m^3}} < y\right\} = P(y)$$

Therefore $N(t)$ is asymptotically Normal with mean t/m and variance $\sigma^2 t/m^3$.

For Poisson process $E[N(t)] = \lambda t = t/m$ and $V[N(t)] = \lambda t = t/m$; for the exponential distribution, $m = 1/\lambda,\;\; \sigma^2 = 1/\lambda^2$

$\dfrac{\sigma^2 t}{m^3} = \dfrac{(1/\lambda)^2 t}{(1/\lambda)^3} = \lambda t.$ Thus the results hold exactly for a Poisson Process.

5.2 Renewal Function

Consider $H(t) = E[N(t)]$.

$H(t)$ is called Renewal Function. If $p_n(t) = P\{N(t) = n\}$, then

$$H(t) = \sum_{n=0}^{\infty} n p_n(t).$$

Taking Laplace Transforms

$$H^*(s) = \sum_{n=0}^{\infty} n p_n^*(s), \quad p_n^*(s) = \int_0^{\infty} e^{-st} p_n(t) dt$$

Recall

$$p_n^*(s) = \frac{q^*(s)^n - q^*(s)^{n+1}}{s}$$

$$H^*(s) = \frac{1}{s}\left\{\sum_1^\infty nq^*(s)^n - \sum_1^\infty nq^*(s)^{n+1}\right\}$$

$$= \frac{1}{s}\left\{q^*(s) + q^*(s)^2 + q^*(s)^3 + \ldots\right\}$$

$$\boxed{H^*(s) \;=\; q^*(s)/s(1-q^*(s))}$$

Note that if $F_n(t) = P\{S_n < t\}$ where $S_n =$ time to n^{th} event, then since $S_n = T_1 + T_2 + \ldots + T_n$, $F_n^*(s) = \dfrac{q_n^*(s)}{s} = \dfrac{q^*(s)^n}{s}$.

Therefore $\boxed{H(t) = \sum_{n=1}^{\infty} F_n(t)}$

Suppose

$$q^*(s) = \lambda/\lambda + s$$

$$H^*(s) = q^*(s)/s(1 - q^*(s))$$

$$H^*(s) = \frac{\lambda/\lambda + s}{s(1 - \dfrac{\lambda}{\lambda + s})} = \frac{\lambda}{s^2}$$

$$\Rightarrow H(t) = E[N(t)] = \lambda t \qquad \text{as } \mathcal{L}^{-1}(\frac{1}{s^2}) = t$$

We will show

$$\boxed{H(t) = \mathcal{L}^{-1}\{H^*(s)\} = \frac{t}{m} + \frac{\sigma^2 - m^2}{2m^2} + o(1)}$$

In general $q^*(s) = E(e^{-st}) = 1 - sm + \dfrac{s^2 m_2}{2} + O(s^3)$

Substituting $q^*(s)$ in $H^*(s)$

$$H^*(s) = \frac{1}{s^2 m} + \frac{\sigma^2 - m^2}{2m^2 s} + O(1)$$

<u>Note:</u> Therefore taking inverse Laplace Transforms of $H^*(s)$ results in

$$H(t) = \frac{t}{m} + \frac{\sigma^2 - m^2}{2m^2} + o(1)$$

$o(1)$ means that $o(1)$ refers to a function $f(t)$ such that

$\lim_{t\to\infty} \dfrac{f(t)}{1} = 0,$ i.e. $f(t) = K/t^n \quad n \geq 1.$

5.3 Renewal Density Function and Related Theorems

Consider $H(t)/t \cong \dfrac{1}{m} + \dfrac{(\sigma^2 - m^2)}{2m^2}\dfrac{1}{t}.$

It is clear that
$$\boxed{\lim_{t\to\infty} H(t)/t = \frac{1}{m}}$$

Furthermore,
$$\boxed{\lim_{t\to\infty}[H(t+a) - H(t)] \cong \frac{t+a}{m} - \frac{t}{m} = \frac{a}{m}}$$

The above result is often referred to as Blackwell's Theorem.

The derivative of $H(t)$; i.e. $h(t) = H'(t)$ is called the renewal density function.

Hence
$$h(t) = \lim_{a\to 0}[H(t+a) - H(t)]/a$$

and
$$\boxed{\lim_{t\to\infty} h(t) = 1/m}$$

The operable definition of $h(t)dt$ is that it is the expected number of events in the interval $(t, t+dt)$ or equivalently the probability of a renewal in $(t, t+dt)$. Therefore for large t, it is a constant and is equivalent to a Poison Process.

Renewal Equation

Recall that

$$h^*(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{H'(t)\} = sH^*(s) - H(0).$$

Since $E[N(0)] = H(0) = 0$, we have

$$h^*(s) = s\left[\frac{q^*(s)}{s(1-q^*(s))}\right] = \frac{q^*(s)}{1-q^*(s)}$$

Hence we can write

$$h^*(s) = q^*(s) + q^*(s)h^*(s)$$

and on taking the inverse transform we have

$$\boxed{h(t) = q(t) + \int_0^t q(\tau)h(t-\tau)d\tau}$$

The interpretation of the above integral equation is that an event takes place in $(t, t+dt)$ with probability $h(t)dt$. It could have been the first event which has probability $q(t)dt$ or a later event. In this latter case the event preceeding the one in $(t, t+dt)$ took place in $(t-\tau, t-\tau+dt)$ with probability $h(t-\tau)dt$ and the time to the next event is $\tau < T \leq \tau + d\tau$ with probability $q(\tau)d\tau$. Integrating over all possible values of τ $(0$ to $t)$ gives the above integral expression.

Consider the expression $W(t) = \int_0^t w(\tau)h(\tau)d\tau$

where $w(t)$ is a non-negative function such that $\int_0^\infty w(\tau)d\tau < \infty$.

Then as $t \to \infty$

$$\boxed{\lim_{t\to\infty} W(t) = \frac{1}{m}\int_0^\infty w(\tau)d\tau}$$

The above is often called the key renewal theorem. An interpretation of $\lim_{t\to\infty} W(t)$ is that it is the expected value of a random variable(for large t) in which a value $w(\tau)$ is observed at every event. For example, if the event is an earthquake, the w may refer to the magnitude of the earthquake on the Richter scale.

A more realistic application of the key renewal theorem is to approximate $W(t+a) - W(t)$ for large t by

$$W(t+a) - W(t) \cong \frac{1}{m}\int_t^{t+a} w(\tau)d\tau.$$

5.4 Equilibrium Renewal Process

Suppose a renewal process is going on for a long time. It starts to be observed at a point in chronological time which is designated as time 0. Define T_1 to be the time to the first event after time 0. It has a forward recurrence time distribution $q_f(t) = Q(t)/m$.

Then $S_n = T_1 + T_2 + ... + T_n$ has a pdf $f_n(t)$ having the Laplace transform

$$f_n^*(s) = q_f^*(s) q^*(s)^{n-1} = \frac{Q^*(s) q^*(s)^{n-1}}{m}$$

$$= \frac{[1 - q^*(s)] {q^*}^{n-1}(s)}{sm}$$

This process is called an equilibrium renewal process.

Since

$$p_n^*(s) = \frac{f_n^*(s) - f_{n+1}^*(s)}{s}$$
$$= \frac{[1 - q^*(s)][q^*(s)^{n-1} - q^*(s)^n]}{s^2 m}$$

$$H_e(t) = E[N(t)] = \sum_{n=1}^{\infty} n p_n(t)$$

$$H_e^*(s) = \sum_{1}^{\infty} n p_n^*(s) = \left[\frac{1 - q^*(s)}{s^2 m}\right][1 + q^*(s) + q^*(s^2) + \ldots]$$
$$= \frac{1 - q^*(s)}{s^2 m} \frac{1}{1 - q^*(s)} = \frac{1}{s^2 m}$$

and $$\boxed{H_e(t) = E(N(t)) = t/m}$$

Note: $$H_e(t_2) - H_e(t_1) = \frac{t_2 - t_1}{m} = H_e(t_2 - t_1)$$

5.5 Appendix: Notes on Asymptotic Relations

1. Big "O" $f(x) = O[g(x)]$

$f(x)$ is of the order of $g(x)$ as $x \to a$ iff $\lim_{x \to a} \frac{f(x)}{g(x)} < \infty$ (bounded)

2. Little "o" $f(x) = o[g(x)]$

$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$, $f(x)$ becomes negligible compared with $g(x)$ as $x \to a$

3. $\sim$ $f(x) \sim g(x)$

$f(x)$ is asymptotically proportional to $g(x)$ as $x \to a$

iff $\lim_{x \to a} \frac{f(x)}{g(x)} < \infty$ and $\neq 0$

4. $\simeq$ $f(x) \simeq g(x)$,

$f(x)$ is asymptotically = $g(x)$ iff $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$